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ON THE SPECTRA OF THE RESTRICTIONS OF AN OPERATOR

BY

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ABSTRACT. Let T be a bounded linear operator from a complex Banach space $\mathfrak{X}$ into itself and let $\mathfrak{M}$ be a closed invariant subspace of T. Let $T|\mathfrak{M}$ denote the restriction of T to $\mathfrak{M}$ and let σ denote the spectrum of an operator. The main results say that: (1) If $\mathfrak{X}$ is the closed linear span of a family $\{\mathfrak{M}_\nu\}$ of invariant subspaces, then every component of $\sigma(T)$ intersects the closure of the set $\bigcup_\nu \sigma(T|\mathfrak{M}_\nu)$ and every point of $\sigma(T)\setminus \bigcup_\nu \sigma(T|\mathfrak{M}_\nu)$ is an approximate eigenvalue of T. (2) If $\mathfrak{X}$ is the closed linear span of a finite family $\{\mathfrak{M}_1, \ldots, \mathfrak{M}_n\}$ of invariant subspaces, and the spectra $\sigma(T|\mathfrak{M}_j)$, $j = 1, 2, \ldots, n$, are pairwise disjoint, then $\mathfrak{X}$ is actually equal to the algebraic direct sum of the $\mathfrak{M}_j$'s, the $\mathfrak{M}_j$'s are hyperinvariant subspaces of T and $\sigma(T) = \bigcup_{j=1}^n \sigma(T|\mathfrak{M}_j)$. This last result is sharp in a certain specified sense. The results of (1) have a "dual version" (1'); (1) and (1') are applied to analyze the spectrum of an operator having a chain of invariant subspaces which is "piecewise well-ordered by inclusion", extending in several ways recent results of J. D. Stafney on the spectra of lower triangular matrices.

1. On the spectra of the restrictions of $T \in \mathfrak{L}(\mathfrak{X})$. In what follows $\mathfrak{X}$ will denote a Banach space over the complex field $\mathbf{C}$: *Operator* and *subspace* will mean *bounded linear map from* $\mathfrak{X}$ *into* $\mathfrak{X}$ and *closed linear manifold*, respectively. The Banach algebra of all operators acting on $\mathfrak{X}$ will be denoted by $\mathfrak{L}(\mathfrak{X})$. Let $\mathfrak{M} \in \operatorname{Lat} T$, the lattice of invariant subspaces of an operator T. Here we shall study the relations between the spectrum of T and the spectra of $T|\mathfrak{M}$ (the restriction of T to $\mathfrak{M}$ thought as an operator acting on $\mathfrak{M}$) and $\bar{T}_{\mathfrak{M}} \in \mathfrak{L}(\mathfrak{X}/\mathfrak{M})$, the operator induced by T on the quotient space $\mathfrak{X}/\mathfrak{M}$, which is defined by the equality $\bar{T}\pi x = \pi T x$, for all $x \in \mathfrak{X}$, where π: $\mathfrak{X} \to \mathfrak{X}/\mathfrak{M}$ is the canonical projection.

Let $\sigma(T)$ be the spectrum of $T \in \mathfrak{L}(\mathfrak{X})$; following P. R. Halmos [3], we shall denote by $\sigma_{\mathrm{ap}}(T)$ the *approximate point spectrum* of T, i.e. $\sigma_{\mathrm{ap}}(T) = \{\lambda \in \sigma(T)$: there exists a sequence $(x_n)_{n=1}^\infty$ of vectors of norm 1 such that

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$\|(T-\lambda)x_n\| \to 0$, as $n \to \infty\}$. It is well known that $\sigma_{ap}(T)$ contains $\sigma_p(T)$, the point spectrum of T, $\sigma_{ap}(T)$ is a closed subset of $\sigma(T)$, $\sigma_{ap}(T) \supset \partial\sigma(T)$, (where ∂K denotes the boundary of a set $K \subset \mathbf{C}$) and $(T-\lambda)$ is a semi-Fredholm operator of negative index for all $\lambda \in \sigma(T) \setminus \sigma_{ap}(T)$. If $\mathfrak{M} \in \operatorname{Lat} T$, then $\partial\sigma(T|\mathfrak{M}) \subset \sigma_{ap}(T|\mathfrak{M}) \subset \sigma_{ap}(T)$; in particular, $\sigma(T|\mathfrak{M}) \cap \sigma(T) \neq \varnothing$, unless $\mathfrak{M} = \{0\}$ (in which case $\sigma(T|\mathfrak{M}) = \varnothing$). If $T^* \in \mathcal{L}(\mathfrak{X}^*)$ denotes the adjoint of T (acting on the dual space $\mathfrak{X}^*$ of $\mathfrak{X}$), then $\sigma(T^*) = \sigma(T)$ (except when $\mathfrak{X}$ is a Hilbert space and T^* is defined "via inner product", in which case the above equality is replaced by $\sigma(T^*) = \{\bar{\lambda}: \lambda \in \sigma(T)\}$). The reader is referred to [1], [3], [9], [10] for details.

A fundamental tool for this paper is the *Riesz functional calculus*: if f is a function analytic in a neighborhood Ω of $\sigma(T)$ and γ is a suitably oriented finite family of rectifiable closed pairwise disjoint Jordan curves such that $(1/2\pi i)\int_\gamma (\lambda - z)^{-1}\, d\lambda = 1$ for all $z \in \sigma(T)$, then $f(T) = (1/2\pi i)\int_\gamma f(\lambda)(\lambda - T)^{-1}\, d\lambda$ defines an operator on $\mathfrak{X}$ which can be approximated in the norm by rational function of T, with poles outside $\sigma(T)$; then $f(T) \in \mathcal{Q}_T^a$, the *analytic algebra generated by* T. Moreover, if f is actually analytic in a neighborhood of $\hat{\sigma}(T) = \mathbf{C} \setminus \rho_\infty(T)$, where $\rho_\infty(T)$ denotes the unbounded component of the resolvent set $\rho(T) = \mathbf{C} \setminus \sigma(T)$ of T, then $f(T) \in \mathcal{Q}_T$, the weak closure of the polynomials in T. (This is an easy consequence of Runge's theorem; see [6]. The Riesz' functional calculus that we need is contained in [9] and [10, Chapter XI].) We shall extensively use a particular case of this functional calculus; namely, if $\sigma(T) = \sigma_0 \cup \sigma_1$, where σ_0 and σ_1 are nonempty disjoint *clopen* (i.e., closed and open) subsets of $\sigma(T)$ and f is defined to be identically zero in a neighborhood of σ_0 and identically one in a neighborhood of σ_1, then $f(T)$ is an idempotent element of $\mathcal{Q}_T^a$ such that $\mathfrak{X} = \ker f(T) \oplus \operatorname{ran} f(T)$ (the *algebraic* direct sum of the kernel and the range of $f(T)$), $\sigma(T|\ker f(T)) = \sigma_0$ and $\sigma(T|\operatorname{ran} f(T)) = \sigma_1$. $f(T)$ will be called the idempotent associated to σ_1.

THEOREM 1. *Let $\{\mathfrak{M}_\nu: \nu \in \Phi\}$ be an arbitrary family of invariant subspaces of the operator T whose closed linear span $\bigvee\{\mathfrak{M}_\nu: \nu \in \Phi\}$ is the whole space $\mathfrak{X}$. Let $\sigma' = \bigcup_\nu \sigma(T_\nu)$, where $T_\nu = T|\mathfrak{M}_\nu$ is the restriction of T to $\mathfrak{M}_\nu$, and $\sigma =$* closure σ'. *Then*:

(i) $\sigma(T) \setminus \sigma' \subset \sigma_{ap}(T)$.

(ii) *Every clopen subset of $\sigma(T)$ intersects σ'.*

(iii) *Every component of $\sigma(T)$ intersects σ.*

(iv) *In particular, if $\sigma(T)$ is totally disconnected, then $\sigma(T) = \sigma$.*

PROOF. (i) We proceed as in [5, Theorem 6]: if $\lambda \in \sigma(T) \setminus \sigma'$ then $(T - \lambda)\mathfrak{M}_\nu = \mathfrak{M}_\nu$ for all $\nu \in \Phi$ and, therefore,

$$(T-\lambda)\mathfrak{X} = (T-\lambda)\bigvee_\nu \mathfrak{M}_\nu \supset (T-\lambda)\sum_\nu \mathfrak{M}_\nu = \sum_\nu (T-\lambda)\mathfrak{M}_\nu = \sum_\nu \mathfrak{M}_\nu,$$

which is dense in $\mathfrak{X}$, i.e. $T - \lambda$ has a dense range, whence the result follows (see [9, Chapter IV]).

(ii) Let σ_0 be a clopen subset of $\sigma(T)$ which does not intersect σ'. For fixed ν, let f_ν be a function analytic in a neighborhood of $\sigma(T) \cup \sigma(T_\nu)$ such that $f(z) \equiv 0$ in a neighborhood of σ_0 and $f(z) \equiv 1$ in a neighborhood of $\sigma(T) \cup \sigma(T_\nu) \setminus \sigma_0$, and let $E_\nu = f_\nu(T)$. Observe that $E = E_\nu$ is an idempotent independent of $\nu \in \Phi$ [10, Chapter XI]; in fact, for any ν, $\mathfrak{X} = \operatorname{ran} E_\nu \oplus \ker E_\nu$ is the decomposition of $\mathfrak{X}$ associated to the partition $\sigma(T) = [\sigma(T) \setminus \sigma_0] \cup \sigma_0$, where $\sigma(T|\operatorname{ran} E_\nu) = \sigma(T) \setminus \sigma_0$ and $\sigma(T|\ker E_\nu) = \sigma_0$, $\operatorname{ran} E_\nu$ and $\ker E_\nu$ are obviously invariant under E_ν, and $E_\nu|\operatorname{ran} E_\nu = I|\operatorname{ran} E_\nu$ (where I denotes the identity operator on $\mathfrak{X}$) and $E_\nu|\ker E_\nu = 0$. Clearly, the above partition does not depend on ν.

On the other hand, $\mathfrak{M}_\nu$ is invariant under E and $E|\mathfrak{M}_\nu = f_\nu(T)|\mathfrak{M}_\nu = f_\nu(T|\mathfrak{M}_\nu) = I|\mathfrak{M}_\nu$ because $f_\nu(z) \equiv 1$ in a neighborhood of $\sigma(T_\nu)$. Since $\mathfrak{X} = \bigvee_\nu \mathfrak{M}_\nu$, it follows that $E = I$; i.e. $\ker E = \{0\}$, which is impossible unless $\sigma_0 = \emptyset$. This proves (ii).

(iii) Since $\sigma(T)$ is a compact Hausdorff space, every component of $\sigma(T)$ is the intersection of all the clopen subsets containing it. Let $\sigma_1 = \bigcap \{\sigma_\alpha$: $(\alpha \in \Psi)\sigma_\alpha$ is clopen and contains $\sigma_1\}$ be a component of $\sigma(T)$. According to (ii), $\sigma_\alpha \cap \sigma' \neq \emptyset$ for every $\alpha \in \Psi$. Therefore $\{\sigma_\alpha \cap \sigma\colon \alpha \in \Psi\}$ is a family of closed subsets of $\sigma(T)$ having the finite intersection property. By compactness, it is clear that $\sigma_1 \cap \sigma = \bigcap \{\sigma_\alpha \cap \sigma\colon \alpha \in \Psi\}$ cannot be empty.

(iv) The proof of (ii) shows that if $\sigma(T)$ is totally disconnected, then $\sigma(T) \subset \sigma$. Now, according to [1], for each $\nu \in \Phi$, $\sigma(T_\nu) \subset \hat{\sigma}(T)$. Since $\sigma(T)$ is totally disconnected, $\sigma(T) = \hat{\sigma}(T)$ and therefore $\sigma(T_\nu) \subset \sigma(T)$; a fortiori, $\sigma \subset \sigma(T)$. $\square$

Example C of [5] shows that, in general, $\sigma(T) \not\supset \sigma$. Our next example shows that $\sigma(T) \not\subset \hat{\sigma}$ in the general case; moreover, it also shows that σ could be surprisingly small in comparison with $\sigma(T)$, even in the case when $\mathfrak{X}$ is the closed span of only two subspaces. The ingredients for the construction of this example were taken from a paper of T. B. Hoover [8].

Example E. Let $\mathfrak{X} = \mathfrak{X}_0 \oplus \mathfrak{X}_1$, where $\mathfrak{X}_0$ and $\mathfrak{X}_1$ are separable infinite dimensional Hilbert spaces. We can write $\mathfrak{X}_0 = \bigoplus_{n+1}^\infty \mathfrak{X}_{0(n)}$, $\mathfrak{X}_1 = \bigoplus_{n=1}^\infty \mathfrak{X}_{1(n)}$ (all the direct sums considered here are closed orthogonal direct sums in Hilbert spaces), where $\mathfrak{X}_{0(n)} \cong \mathbf{C}^n \cong \mathfrak{X}_{1(n)}$, for $n = 1, 2, \ldots$. Let $\{x_1^n, \ldots, x_n^n\}$ and $\{y_1^n, \ldots, y_n^n\}$ be orthonormal bases of $\mathfrak{X}_{0(n)}$ and $\mathfrak{X}_{1(n)}$, resp. ($n = 1, 2, \ldots$) and define $T \in \mathfrak{L}(\mathfrak{X})$ as follows: $T = T_0 \oplus T_1$, $T_j = \bigoplus_{n=1}^\infty T_{j(n)} \in \mathfrak{L}(\mathfrak{X}_j)$, $j = 0, 1$, where $T_{j(n)}$ ($j = 0, 1$; $n = 1, 2, \ldots$) are the operators acting on $\mathfrak{X}_{j(n)}$ defined by $T_{0(n)}x_1^n = 0$, $T_{0(n)}x_k^n = x_{k-1}^n$, for $k = 2, 3, \ldots, n$, and $T_{1(n)}y_1^n = 0$, $T_{1(n)}y_k^n = (1/n)y_{k-1}^n$, for $k = 2, 3, \ldots, n$.

Let $\mathfrak{X}_2 = \bigoplus_{n=1}^\infty \mathfrak{X}_{2(n)}$, where $\mathfrak{X}_{2(n)}$ is the subspace spanned by the orthonormal set $\{z_k^n = c_{k,n}(n^{-k}x_k^n + y_k^n)\}_{k=1}^n$ $(c_{k,n} = (1 + n^{-2k})^{-1/2})$. It is easy to check that $\mathfrak{X}_{0(n)}$, $\mathfrak{X}_{1(n)}$, $\mathfrak{X}_{2(n)} \in \operatorname{Lat} T_{0(n)} \oplus T_{1(n)}$, and therefore $\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2 \in \operatorname{Lat} T$. Clearly, we have $\mathfrak{X}_{1(n)} + \mathfrak{X}_{2(n)} = \mathfrak{X}_{0(n)} \oplus \mathfrak{X}_{1(n)}$ and, therefore, $\mathfrak{X} = \mathfrak{X}_1 \vee \mathfrak{X}_2$. On the other hand, $\mathfrak{X}_1 \cap \mathfrak{X}_2 = \{0\}$ and $\sigma(T|\mathfrak{X}_1) = \sigma(T|\mathfrak{X}_2) = \{0\}$, while $\sigma(T) = D^-$ (i.e., the closure of the open unit disc; see [3], [8]).

REMARKS. (a) Let $\mathcal{Q}'_T = \{L \in \mathfrak{L}(\mathfrak{X}): LT = TL\}$ be the *commutant* of T in $\mathfrak{L}(X)$ and let $\mathcal{Q}''_T$ (similarly defined) be the commutant of $\mathcal{Q}'_T$, i.e. the *double commutant* of T. Then [6] $\mathcal{Q}_T \subset \mathcal{Q}^a_T \subset \mathcal{Q}''_T \subset \mathcal{Q}'_T$ and the corresponding lattices of invariant subspaces satisfy the reverse inclusions, i.e.: $\operatorname{Lat} \mathcal{Q}_T = \operatorname{Lat} T \supset \operatorname{Lat} \mathcal{Q}^a_T$ (*analytically* invariant subspaces) $\supset \operatorname{Lat} \mathcal{Q}''_T$ (*bi-invariant* subspaces) $\supset \operatorname{Lat} \mathcal{Q}'_T$ (*hyperinvariant* subspaces) (Lat $\mathcal{Q}'_T$ has also been studied in [2]). A straightforward computation shows that the operator of Example E satisfies $\mathcal{Q}_T = \mathcal{Q}''_T$ and, therefore, $\mathfrak{X}_1, \mathfrak{X}_2 \in \operatorname{Lat} \mathcal{Q}''_T$. We do not know any example of that kind of pathological behaviour with *hyperinvariant* $\mathfrak{X}_1$ and $\mathfrak{X}_2$.

(b) The arguments of the proof of Theorem 1 can be used to improve a result of T. B. Hoover [8]: If $A \in \mathfrak{L}(\mathfrak{X})$, $B \in \mathfrak{L}(\mathfrak{Y})$ and there exists a quasi-invertible continuous linear map $S: \mathfrak{X} \to \mathfrak{Y}$ (i.e. ker $S = \{0\}$ and ran S is dense in $\mathfrak{Y}$) such that $SA = BS$, then every component of $\sigma(A)$ intersects some component of $\sigma(B)$ and conversely. In fact, the equality $SA = BS$ implies that $S(A - \lambda)^{-1} = (B - \lambda)^{-1}S$ for all $\lambda \in \rho(A) \cap \rho(B)$; thus, if f is a function analytic in a neighborhood of $\sigma(A) \cup \sigma(B)$ which only takes the values 0 and 1, then $Sf(A) = f(B)S$. Since S is quasi-invertible, we conclude that $f(A) = 0$ if and only $f(B) = 0$, and it clearly follows that every clopen subset of $\sigma(A)$ intersects some clopen subset of $\sigma(B)$. By proceeding as in the *proof* of (iii), it is easy to see that the clopen subsets can be replaced by components of $\sigma(A)$ and $\sigma(B)$. A minor modification of the example given in [8] shows that $\sigma(A)$ can have exactly one component while $\sigma(B)$ has uncountably many.

Our next result is the promised "dual version" of Theorem 1.

THEOREM 1*. *Let $\{\mathfrak{M}_\nu: \nu \in \Phi\}$ be an arbitrary family of invariant subspaces of the operator T such that $\bigcap\{\mathfrak{X}_\nu: \nu \in \Phi\} = \{0\}$. Let $\sigma' = \bigcup_\nu \sigma(\overline{T}_\nu)$, where $\overline{T}_\nu$ is the operator induced by T on $\mathfrak{X}/\mathfrak{M}_\nu$, and $\sigma =$* closure σ'. *Then*:

(i*) $\sigma(T^*) \setminus \sigma' \subset \sigma_{ap}(T^*)$.

(ii*) *Every clopen subset of $\sigma(T)$ intersects σ'.*

(iii*) *Every component of $\sigma(T)$ intersects σ.*

(iv*) *If $\sigma(T)$ is totally disconnected, then $\sigma(T) = \sigma$.*

PROOF. (i*) Let $\mathfrak{M} \in \operatorname{Lat} T$ and let $\mathfrak{M}^\perp = \{x^* \in \mathfrak{X}^*: \ker x^* \supset \mathfrak{M}\}$ be the annihilator of $\mathfrak{M}$. It is well known that $\mathfrak{M}^\perp$ is a w^*-closed subspace of

$\mathfrak{X}^*$ invariant under T^*, which can be canonically identified with $(\mathfrak{X}/\mathfrak{M})^*$; $T^*|\mathfrak{M}^\perp$ can be identified with $\overline{T}^*$ (where, as usual, $\overline{T}$ denotes the operator induced by T on $\mathfrak{X}/\mathfrak{M}$) and, therefore, $\sigma(T^*|\mathfrak{M}^\perp) = \sigma(\overline{T})$ (see [3], [9], [10]).

Since $\bigcap_\nu \mathfrak{M}_\nu = \{0\}$, it follows that the w^*-closed span $w^* - \bigvee_\nu \mathfrak{M}_\nu^\perp$ of the $\mathfrak{M}_\nu^\perp$'s is the whole dual space $\mathfrak{X}^*$. We have shown that $\sigma(T^*) \setminus \bigcup_\nu \sigma(T_\nu^*) = \sigma(T) \setminus \sigma'$ ($T_\nu^* = T^*|\mathfrak{M}_\nu^\perp$). Thus, if $\lambda \in \sigma(T) \setminus \sigma'$ we can proceed as in the proof of (i) to show that $(T^* - \lambda)\mathfrak{X}^* \supset \Sigma_\nu \mathfrak{M}_\nu^\perp$; hence $\operatorname{ran}(T^* - \lambda)$ is w^*-dense in $\mathfrak{X}^*$. If $\operatorname{ran}(T^* - \lambda)$ is *not* closed, then $\lambda \in \sigma_{\rm ap}(T^*)$ because $(T^* - \lambda)$ cannot be bounded below. If $\operatorname{ran}(T^* - \lambda)$ is closed, then $\operatorname{ran}(T^* - \lambda) = [\ker(T - \lambda)]^\perp$ is w^*-closed and, therefore, $\operatorname{ran}(T^* - \lambda) = \mathfrak{X}^*$, $(T^* - \lambda)$ is a semi-Fredholm operator of positive index and $\lambda \in \sigma_p(T^*) \subset \sigma_{\rm ap}(T^*)$ (see [9, Chapter IV]). This proves (i*).

(ii*) Let σ_0 be a clopen subset of $\sigma(T)$ which does not intersect σ'. For fixed ν, let f_ν be defined as in the proof of (ii), with $\sigma(T_\nu)$ replaced by $\sigma(\overline{T}_\nu)$. Let $E_\nu = f_\nu(T)$; then $E_\nu^* = [f_\nu(T)]^* = f_\nu(T^*)$ and $E = E_\nu$, $E^* = E_\nu^*$ are independent of ν. Since f_ν is analytic in a neighborhood of $\sigma(T) \cup \sigma(\overline{T}_\nu)$ and $\overline{T}_\nu - \lambda \in \mathfrak{L}(\mathfrak{X}/\mathfrak{M}_\nu)$ is the operator induced by $T - \lambda$ on $\mathfrak{X}/\mathfrak{M}_\nu$, it follows that $\mathfrak{M}_\nu \in \operatorname{Lat} E_\nu$ and $\overline{E}_\nu = f_\nu(\overline{T}_\nu)$ is equal to the operator induced by $E_\nu = E$ on $\mathfrak{X}/\mathfrak{M}_\nu$. Then $\mathfrak{X} = \operatorname{ran} E \oplus \ker E$ and $\mathfrak{X}^* = \operatorname{ran} E^* \oplus \ker E^*$, $\ker E^* = (\operatorname{ran} E)^\perp$ and $\operatorname{ran} E^* = (\ker E)^\perp$. By using the canonical identification of (i*), it is not difficult to see that $(\ker E_\nu)^\perp \supset (\ker \overline{E}_\nu)^\perp = \mathfrak{M}_\nu^\perp$. Hence, $\operatorname{ran} E^*$ is a w^*-closed subspace of $\mathfrak{X}^*$ containing $\Sigma_\nu \mathfrak{M}_\nu^\perp$, which is a w^*-dense linear manifold; therefore $\operatorname{ran} E^* = \mathfrak{X}^*$, whence we obtain that $E^* = I^*$ and $\sigma_0 = \varnothing$.

The proof of (ii*) is complete. Finally, the proofs of (iii*) and (iv*) are identical to those of (iii) and (iv), resp. □

2. **Triangular operators.** In this section we shall improve and extend several results of J. D. Stafney (see [11]) about the spectra of lower triangular matrices of a certain type. Let $\mathfrak{M}, \mathfrak{N} \in \operatorname{Lat} T$, such that $\mathfrak{M} \subset \mathfrak{N}$ and $\dim \mathfrak{N}/\mathfrak{M} = 1$; then the eigenvalue of the one-dimensional operator induced by $T|\mathfrak{N}$ on $\mathfrak{N}/\mathfrak{M}$ will be called a *diagonal entry* of T.

THEOREM 2. *Let* $T \in \mathfrak{L}(\mathfrak{X})$ *and let* $\mathcal{C}$ *be a chain in* Lat T *such that*:

(1) $\mathcal{C} = \{\mathfrak{M}_\nu: \nu \in \Phi\}$ *is well-ordered from below*; *i.e.*, Φ *is an initial segment of the ordinals and* $\alpha, \beta \in \Phi$, $\alpha \leqslant \beta$ *implies that* $\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$.

(2) $\mathfrak{M}_0 = \{0\}$; *for each* $\nu \in \Phi$, $\dim \mathfrak{M}_{\nu+1}/\mathfrak{M}_\nu = 1$, *and for each limit ordinal* γ, $\mathfrak{M}_\gamma = \bigvee\{\mathfrak{M}_\nu: \nu < \gamma\}$.

(3) $\mathfrak{X} = \bigvee\{\mathfrak{M}_\nu: \nu \in \Phi\}$.

Then: (i) $\mathcal{C} \subset \operatorname{Lat} \mathcal{A}_T^a$; $\{\sigma(T_\nu): \nu \in \Phi\}$ *is an increasing family of compact subsets of* $\sigma(T)$ *and* $\sigma(T) = \sigma_{\rm ap}(T)$ ($T_\nu = T|\mathfrak{M}_\nu$).

(ii) *Every clopen subset of* $\sigma(T)$ *intersects* $d(T) = \{\lambda_\nu : \nu \in \Phi\}$, *where* λ_ν *is the diagonal entry of* T *corresponding to* $[\mathfrak{M}_\nu, \mathfrak{M}_{\nu+1}]$, *and every component of* $\sigma(T)$ *intersects* $d(T)^-$, *the closure of* $d(T)$. *In particular, every isolated point of* $\sigma(T)$ *is a diagonal entry; moreover, these isolated points are eigenvalues of* T.

(iii) *For each* $\nu \in \Phi$, $\sigma(T) = \sigma(T_\nu) \cup \sigma(\bar{T}_\nu)$, *where* $\bar{T}_\nu$ *is the operator induced by* T *on* $\mathfrak{X}/\mathfrak{M}_\nu$, *and* $\{\sigma(\bar{T}_\nu): \nu \in \Phi\}$ *is a decreasing family of compact subsets of* $\sigma(T)$.

(iv) *Let* $\bar{\sigma}_f(T) = \bigcap \{\sigma(\bar{T}_\nu): \nu \in \Phi\}$. *If* $\Phi = \{0, 1, 2, \dots\}$, *then* $\sigma(T) = \bar{\sigma}_f(T) \cup d(T)$.

PROOF. (i) Let $\Phi = \{0, 1, 2, \dots, \omega, \omega + 1, \dots, \nu, \nu + 1, \dots\}$. For each finite n, $\mathfrak{M}_n \in \operatorname{Lat} \mathfrak{A}_T^a$ and $\sigma(T_n) = \{\lambda_0, \lambda_1, \dots, \lambda_{n-1}\}$ (or $\varnothing$, if $n = 0$; see [5, Theorem 8]) and $\{\sigma(T_n): n = 0, 1, 2, \dots\}$ is an increasing sequence of finite subsets of $\sigma(T)$.

Now we proceed by transfinite induction: if $\mathfrak{M}_\nu \in \operatorname{Lat} \mathfrak{A}_T^a$ and $\sigma(T_\nu) = \sigma_{\mathrm{ap}}(T_\nu)$, then $\mathfrak{M}_\nu \in \operatorname{Lat} T_{\nu+1}$ and (since $\dim \mathfrak{M}_{\nu+1}/\mathfrak{M}_\nu = 1$) it follows from [5, Theorem 8] that $\mathfrak{M}_\nu \in \operatorname{Lat} \mathfrak{A}_{T_{\nu+1}}^a$, which is contained in $\operatorname{Lat} \mathfrak{A}_T^a$ [6, Theorem 6.1]; moreover, $\sigma(T_{\nu+1}) = \sigma(T_\nu) \cup \{\lambda_\nu\} = \sigma_{\mathrm{ap}}(T_{\nu+}) \cup \{\lambda_\nu\} = \sigma_{\mathrm{ap}}(T_{\nu+1})$, because $\sigma(T_{\nu+1}) \setminus \sigma_{\mathrm{ap}}(T_{\nu+1})$ is an open subset of $\mathbf{C}$ contained in $\{\lambda_\nu\}$ (in fact, $[\sigma(T_{\nu+1}) \setminus \sigma_{\mathrm{ap}}(T_{\nu+1})] \cap \sigma_{\mathrm{ap}}(T_\nu) = \varnothing$) and therefore it must be empty. If γ is a limit ordinal and $\mathfrak{M}_\nu \in \operatorname{Lat} \mathfrak{A}_T^a$ and $\sigma(T_\nu) = \sigma_{\mathrm{ap}}(T_\nu)$ for all $\nu < \gamma$, then $\mathfrak{M}_\gamma \in \operatorname{Lat} \mathfrak{A}_T^a$ because $\mathfrak{M}_\gamma = \bigvee\{\mathfrak{M}_\nu : \nu < \gamma\}$ and $\operatorname{Lat} \mathfrak{A}_T^a$ is complete; moreover, by Theorem 1(i),

$$\sigma(T_\gamma) = \sigma_{\mathrm{ap}}(T_\gamma) \cup \left\{ \bigcup_{\nu<\gamma} \sigma_{\mathrm{ap}}(T_\nu) \right\} = \sigma_{\mathrm{ap}}(T_\gamma),$$

which is clearly contained in $\sigma(T)$.

We have shown, in particular, that $\{\sigma(T_\nu)\}$ is an increasing family of sets and that $\mathcal{C} \in \operatorname{Lat} \mathfrak{A}_T^a$. Finally, by applying Theorem 1(i) to T and using (3), we conclude that $\sigma(T) = \sigma_{\mathrm{ap}}(T)$.

(ii) Let σ_0 be a nonempty clopen subset of $\sigma(T)$ and let E_0 be the associated idempotent. Then $E_0 \in \mathfrak{A}_T^a$ and $\mathfrak{X} = \operatorname{ran} E_0 \oplus \ker E_0$, where $\operatorname{ran} E_0$, $\ker E_0$ $\in \operatorname{Lat} \mathfrak{A}_T^a$, $\sigma(T|\operatorname{ran} E_0) = \sigma_0$ and $\sigma(T|\ker E_0) \cap \sigma_0 = \varnothing$. $\mathfrak{A}_T^a$ and $\operatorname{Lat} \mathfrak{A}_T^a$ split with respect to the above decomposition of $\mathfrak{X}$ and, therefore (since $\mathcal{C} \subset \operatorname{Lat} \mathfrak{A}_T^a$), $\mathfrak{M}_\nu = (\mathfrak{M}_\nu \cap \operatorname{ran} E_0) \oplus (\mathfrak{M}_\nu \cap \ker E_0)$ for all $\nu \in \Phi$ (see [6]).

Condition (3) guarantees that $E_0\mathfrak{M}_\nu = \mathfrak{M}_\nu \cap \operatorname{ran} E_0 \neq \{0\}$ for all ν in a final segment of Φ. Let γ be the first index such that $E_0\mathfrak{M}_\gamma \neq \{0\}$. Since $\sigma(T_\gamma) \subset \sigma(T)$, $\sigma_0 \cap \sigma(T_\gamma)$ is a nonempty clopen subset of $\sigma(T_\gamma)$ and therefore, by Theorem 1(ii), $\sigma_0 \cap \sigma(T_\gamma)$ must intersect $\bigcup\{\sigma(T_\nu): \nu < \gamma\}$; but this implies that σ_0 intersects $\sigma(T_\nu)$ for some $\nu < \gamma$, contradicting the definition of γ, unless $\gamma = \alpha + 1$ for some ordinal $\alpha \in \Phi$ such that $\sigma_0 \cap \sigma(T_\alpha) = \varnothing$. It follows that $\sigma_0 \cap \sigma(T_\gamma) = \sigma_0 \cap \sigma(T_{\alpha+1}) = \{\lambda_\alpha\}$, i.e. $\lambda_\alpha \in \sigma_0$.

This proves the first part of (ii); the second statement follows from the first one as in the proof of Theorem 1 (iii). The fact that every isolated point of $\sigma(T)$ belongs to $d(T)$ is also clear; thus, it only remains to show that the isolated points are eigenvalues of T. Let E_γ be the idempotent associated to $\{\lambda_\gamma\}$, where $\lambda_\gamma \in d(T)$ is an isolated point of $\sigma(T)$. Without loss of generality we can assume that γ is the first index such that $\lambda_\nu = \lambda_\gamma$. Then $\mathfrak{M}_{\gamma+1} = \mathfrak{M}_\gamma \oplus \{cx_\gamma : c \in \mathbf{C}\}$, where $\mathfrak{M}_\gamma = \mathfrak{M}_{\gamma+1} \cap \ker E_\gamma$, $\sigma(T_\gamma)$ does not contain the point λ_γ and x_γ is any nonzero vector of the one-dimensional subspace $\mathfrak{M}_{\gamma+1} \cap \operatorname{ran} E_\gamma$. It is easy to see that $T_{\gamma+1}x_\gamma = \lambda_\gamma x_\gamma$ and, therefore, $\lambda_\gamma \in \sigma_p(T_{\gamma+1}) \subset \sigma_p(T)$.

(iii) The equality $\sigma(T) = \sigma(T_\nu) \cup \sigma(\bar{T}_\nu)$ follows from (i) and [5, Theorem 3]. As in the proof of (i), we can easily see that $\sigma(\bar{T}_\nu) = \sigma(\bar{T}_{\nu+1}) \cup \{\lambda_\nu\}$ ($\lambda_\nu \in \sigma_p(\bar{T}_\nu)$); hence, $\sigma(\bar{T}_\nu) \supset \sigma(\bar{T}_{\nu+1})$ for all $\nu \in \Phi$. In general, if $\alpha, \beta \in \Phi$ and $\alpha < \beta$, then $\mathfrak{X}/\mathfrak{M}_\beta$ is canonically isomorphic to $(\mathfrak{X}/\mathfrak{M}_\alpha)/(\mathfrak{M}_\beta/\mathfrak{M}_\alpha)$ and, according to [5, Theorem 3], $\sigma(\bar{T}_\alpha) = \sigma(\bar{T}_\beta) \cup \sigma(\bar{T}_\alpha|\mathfrak{M}_\beta/\mathfrak{M}_\alpha) \supset \sigma(\bar{T}_\beta)$, provided $\mathfrak{M}_\beta/\mathfrak{M}_\alpha \in \operatorname{Lat} \mathcal{Q}^a_{\bar{T}_\alpha}$; but this is a consequence of (i). Indeed, it is enough to replace (in the proof of (i)) T by $\bar{T}_\alpha$ and $\mathcal{C}$ by $\{\mathfrak{M}_\nu/\mathfrak{M}_\alpha : \nu \in \Phi, \nu > \alpha\}$. Hence, $\sigma(\bar{T}_\alpha)$ always contains $\sigma(\bar{T}_\beta)$ whenever $\alpha, \beta \in \Phi$ and $\alpha < \beta$. Therefore,

$$\sigma(T) \supset \sigma(\bar{T}_1) \supset \sigma(\bar{T}_2) \supset \cdots \supset \sigma(\bar{T}_\nu) \supset \sigma(\bar{T}_{\nu+1}) \supset \cdots \supset \bar{\sigma}_f(T) = \bigcap_\nu \sigma(\bar{T}_\nu)$$

($\neq \varnothing$, unless $\mathfrak{X} \in \mathcal{C}$).

(iv) Let $\Phi = \{0, 1, 2, \ldots\}$. The proofs of (i) and (iii) show that $\sigma(T) = d(T) \cup \bar{\sigma}_f(T)$. □

Roughly speaking, the operator T of Theorem 2 has an "upper triangular matrix" with respect to the chain $\mathcal{C}$. Analogous results can be proven for an operator having a "lower triangular matrix" with respect to a certain chain of invariant subspaces, by using the fact [5, Theorem 8] that an invariant subspace of finite codimension always belongs to $\operatorname{Lat} \mathcal{Q}^a_T$, and Theorem 1* instead of Theorem 1. Thus, we shall establish here the "dual version" of Theorem 2. The proof is left to the interested reader (if any!).

THEOREM 2*. *Let $T \in \mathcal{L}(\mathfrak{X})$ and let $\mathcal{C}$ be a chain in* Lat T *such that*:

(1) $\mathcal{C} = \{\mathfrak{M}_\nu : \nu \in \Phi\}$ *is well-ordered from above*; *i.e.*, Φ *is an initial segment of the ordinals and* $\alpha, \beta \in \Phi$, $\alpha < \beta$ *implies that* $\mathfrak{M}_\alpha \supset \mathfrak{M}_\beta$.

(2) $\mathfrak{M}_0 = \mathfrak{X}$; *for each* $\nu \in \Phi$, $\dim \mathfrak{M}_\nu/\mathfrak{M}_{\nu+1} = 1$ *and for each limit ordinal* γ, $\mathfrak{M}_\gamma = \bigcap\{\mathfrak{M}_\nu : \nu < \gamma\}$.

(3) $\{0\} = \bigcap\{\mathfrak{M}_\nu : \nu \in \Phi\}$.

Then: (i*) $\mathcal{C} \subset \operatorname{Lat} \mathcal{Q}^a_T$; $\{\sigma(\bar{T}_\nu) : \nu \in \Phi\}$ *is an increasing family of compact* *subsets of* $\sigma(T)$ *and* $\sigma(T^*) = \sigma_{\mathrm{ap}}(T^*)$ (T_ν *and* $\overline{T}_\nu$ *are defined exactly as in Theorem* 2).

(ii*) *Every clopen subset of* $\sigma(T)$ *intersects* $d(T)$ *and every component of* $\sigma(T)$ *intersects* $d(T)^-$. *In particular, every isolated point of* $\sigma(T)$ *is a diagonal entry; moreover, these isolated points are eigenvalues of* T^*.

(iii*) *For each* $\nu \in \Phi$, $\sigma(T) = \sigma(\overline{T}_\nu) \cup \sigma(T_\nu)$ *and* $\{\sigma(T_\nu)\colon \nu \in \Phi\}$ *is a decreasing family of compact subsets of* $\sigma(T)$.

(iv*) *Let* $\sigma_f(T) = \bigcap\{\sigma(T_\nu)\colon \nu \in \Phi\}$. *If* $\Phi = \{0, 1, 2, \ldots\}$, *then* $\sigma(T) = \sigma_f(T) \cup d(T)$.

None of the inclusions $\sigma_p(T) \subset d(T)$ and $d(T) \subset \sigma_p(T)$ is true in general. Indeed, we have the following counterexamples:

EXAMPLE F. Let S be the unilateral shift "multiplication by e^{ix}" in the Hardy space $H^2(D) = \bigvee\{e^{inx}\}_{n=0}^{\infty}$ (the e^{inx}'s form an orthonormal basis of this Hilbert space). Then $\{\mathfrak{M}_n = \bigvee(e^{iky})_{k=0}^{n}\colon n = 0, 1, \ldots\}$ is an invariant subspace chain for S^* satisfying the conditions (1), (2), (3) and (iv) of Theorem 2 and $d(S^*) = \{0\} = d(S^*)^-$. On the other hand, $\sigma_p(S^*) = D$ (see [3], [4]).

EXAMPLE G. Let $T_0 \in \mathfrak{L}(\mathfrak{X}_0)$ be the operator defined in Example E and let $\mathfrak{X} = \mathfrak{X}_0 \oplus \mathbf{C}$. The structure of T_0 makes it clear that T_0 has an invariant subspace chain $\mathcal{C}_0$ satisfying (1), (2), (3) and (iv) of Theorem 2. Theorem 1(i) and the fact that T_0 is unitarily equivalent to its adjoint implies (after some computations, see [3], [8]) that $\sigma_p(T_0) = \sigma_p(T_0^*) = \{0\}$ and $\sigma_{\mathrm{ap}}(T_0) = \sigma_{\mathrm{ap}}(T_0^*) = D^-$. Therefore, for every $\lambda \in D^- \setminus \{0\}$, $\mathrm{ran}(T_0 - \lambda)$ is a *proper* dense linear manifold of $\mathfrak{X}_0$ and $\ker(T_0 - \lambda) = \{0\}$. Fix λ and choose $x_\lambda \in \mathfrak{X}_0 \setminus \mathrm{ran}(T_0 - \lambda)$; then define $T \in \mathfrak{L}(\mathfrak{X})$ by means of the matrix

$$T = \begin{pmatrix} T_0 & A_\lambda \\ 0 & \lambda \end{pmatrix}$$

(acting in the usual fashion on $\mathfrak{X} = \mathfrak{X}_0 \oplus \mathbf{C}$) where $A_\lambda(1) = x_\lambda$. Then $\mathcal{C} = \mathcal{C}_0 \cup \{\mathfrak{X}_0 \oplus (0), \mathfrak{X}\}$ is a chain of invariant subspaces of T satisfying (1), (2) and (3) (the type of order being equal to $\omega + 2$) and $d(T) = \{0, \lambda\}$. However, a straightforward computation shows that $\ker(T - \lambda) = \{0\}$. Indeed, $\sigma_p(T) = \{0\}$.

Let $\{\lambda_\alpha\colon \alpha \in \Xi\}$ be an arbitrary subset of D^-, well-ordered according to the index set Ξ. For each $\alpha \in \Xi$, define T_α as the operator T given above, where $x_\alpha \in \mathfrak{X}_0 \setminus \mathrm{ran}(T_0 - \lambda_\alpha)$ is chosen to be a norm-one vector (hence, $\|T_\alpha\|$ is a constant independent of α), acting on the Hilbert space $\mathfrak{X}_\alpha$ (a copy of the above $\mathfrak{X}$), and let $\mathcal{C}_\alpha$ be the copy of the above $\mathcal{C}$. Finally, define $\mathfrak{Y}$ to be the orthogonal direct sum of the $\mathfrak{X}_\alpha$'s and $L = \bigoplus_\alpha T_\alpha$; clearly, $\|L\| = \|T_\alpha\| < \infty$ and, therefore, $L \in \mathfrak{L}(Y)$, and it is easy to see that $\mathfrak{D} = \times_\alpha \mathcal{C}_\alpha$ (lexicographically ordered) is a chain for L satisfying (1), (2) and (3) of

Theorem 1, $d(D) = \{\lambda_\alpha : \alpha \in \Xi\} \cup \{0\}$ and we still have $\sigma_p(L) = \{0\}$!

The results of Theorems 2 and 2* can be only partially extended to other lattices. Let Λ be a subset of $\mathbf{C}$ containing at most one point of each bounded component of $\rho(T)$ and let $\mathcal{A}_T(\Lambda)$ denote the weakly closed algebra generated by T and $\{(T-\lambda)^{-1} : \lambda \in \Lambda\}$. If $\Omega = \rho_\infty(T) \cup [\bigcup\{\rho_\lambda(T) : \lambda \in \Lambda\}]$, where $\rho_\lambda(T)$ is the component of $\rho(T)$ containing λ, then we shall write $\sigma_\Lambda(T) = \hat{\sigma}(T) \setminus \Omega$. Then, as a corollary of Theorem 7 of [5], we have

Corollary 3. *Let* $\mathcal{C} = \{\mathfrak{M}_\nu : \nu \in \Phi\}$ (*where the index set* Φ *is totally ordered from below by inclusion of the corresponding subspaces*) *be a chain in* Lat $\mathcal{A}_T(\Lambda)$. *Then* $\sigma_\Lambda(T) = \sigma_\Lambda(T_\nu) \cup \sigma_\Lambda(\overline{T}_\nu)$ *for all* $\nu \in \Phi$ *and* $\{\sigma_\Lambda(T_\nu) : \nu \in \Phi\}$ ($\{\sigma_\Lambda(\overline{T}_\nu) : \nu \in \Phi\}$, *resp.*) *is an increasing* (*decreasing, resp.*) *family of compact subsets of* $\sigma_\Lambda(T)$.

This corollary cannot be improved in general, because the "holes" of $\sigma(T_\nu)$ can suddenly disappear if ν is a limit point of the index set. Namely, we have

Example H. Let T be the bilateral shift in $L^2(\partial D, dm)$, as in [5, Example A] (see also [3], [4]) and let $L = (T+2) \oplus (T-2) \in \mathcal{L}(L^2 \oplus L^2)$. Define $\mathcal{C}$ as follows: if $\mathfrak{M}_n = \bigvee\{e^{ikx}\}_{k=n}^{\infty}$, $n = 0, \pm 1, \pm 2, \ldots,$ then $\mathcal{C} = \{\mathfrak{M}_n \oplus (0) = \mathfrak{M}_n^-\} \cup \{L^2 \oplus (0)\} \cup \{L^2 \oplus \mathfrak{M}_n^+\}$. It is easy to see that $\mathcal{C}$ is a chain of invariant subspaces of L, $L^2 \oplus L^2 = \bigvee\{\mathfrak{M} \in \mathcal{C}\}$, $\{0\} = \bigcap\{\mathfrak{M} \in \mathcal{C}\}$, $\sigma(L|\mathfrak{M}_n)$ is a closed disc of radius one centered at (-2) and $\sigma(L|\mathfrak{M}_n^+)$ is the union of $\partial\sigma(L|\mathfrak{M}_n^-)$ with a closed disc of radius one centered at 2, while $\sigma(L|L^2 \oplus \{0\})$ is just the circle of radius one and center (-2). This example also shows that, in general, a component of $\sigma(L)$ need not contain a diagonal entry.

Lemma 4. *Let* $T \in \mathcal{L}(\mathfrak{X})$ *and let* $\mathcal{C} \subset$ Lat $\mathcal{A}_T(\Lambda)$ *be a maximal chain of subspaces of* $\mathfrak{X}$. *Let* σ_0 *be a nonempty clopen subset of* $\sigma_\Lambda(T)$ *with associated idempotent* E_0. *If* T_0 *denotes the restriction of* T *to* ran E_0, *then* $\mathcal{C}_0 = \{\mathfrak{M}_\nu \cap \text{ran}\, E_0 : \nu \in \Phi\}$ *is a maximal chain of subspaces of* ran E_0 *contained in* Lat $\mathcal{A}_{T_0}(\Lambda)$ *and every diagonal entry of* T_0 (*with respect to the chain* $\mathcal{C}_0$) *is also a diagonal entry of* T.

Proof. A maximal chain of subspaces can be characterized by the following two properties: $\mathcal{C}$ is complete and, whenever $\mathfrak{M}, \mathfrak{N} \in \mathcal{C}$, $\mathfrak{M} \subsetneq \mathfrak{N}$ and there is no $\mathfrak{M}' \in \mathcal{C}$ such that $\mathfrak{M} \subsetneq \mathfrak{M}' \subsetneq \mathfrak{N}$, then $\dim \mathfrak{N}/\mathfrak{M} = 1$. On the other hand, it is clear that $E_0 \in \mathcal{A}_T(\Lambda)$ and, therefore, Lat $\mathcal{A}_T(\Lambda)$ splits with respect to the decomposition $\mathfrak{X} = \ker E_0 \oplus \text{ran}\, E_0$ [5], [6]. It follows that $\mathcal{C}_0$ is a maximal chain of subspaces of ran E_0 contained in Lat $\mathcal{A}_{T_0}(\Lambda)$.

Let $\mathfrak{M}, \mathfrak{N} \in \mathcal{C}_0$ be a pair of subspaces such that $\mathfrak{M} \subsetneq \mathfrak{N}$ and $\dim \mathfrak{N}/\mathfrak{M} = 1$, and let λ be the corresponding diagonal entry. Define $\mathfrak{M}' = \bigvee\{\mathfrak{M}_\nu \in \mathcal{C} : \mathfrak{M}_\nu \subset \mathfrak{M} \oplus \ker E_0\}$ and $\mathfrak{N}' = \bigcap\{\mathfrak{M}_\nu \in \mathcal{C} : \mathfrak{M}_\nu \supset$

$\mathfrak{N} \oplus \ker E_0\}$. Clearly, $\mathfrak{M}', \mathfrak{N}' \in \mathcal{C}$ and $\mathfrak{M}' \subsetneq \mathfrak{N}'$; moreover, the maximality of $\mathcal{C}$ implies that $\dim \mathfrak{N}'/\mathfrak{M}' = 1$. Therefore, there exists $\mathfrak{M}'' \subset \ker E_0$ such that $\mathfrak{M}' = \mathfrak{M} \oplus \mathfrak{M}''$ and $\mathfrak{N}' = \mathfrak{N} \oplus \mathfrak{M}''$ (to see this, use the fact that $\mathcal{C}$ splits), whence it readily follows that the diagonal entry of T corresponding to the pair $\mathfrak{M}', \mathfrak{N}'$ is the above λ. □

COROLLARY 5. *Let $T \in \mathcal{L}(\mathfrak{X})$ and let $\mathcal{C} \subset \operatorname{Lat} \mathcal{A}_T(\Lambda)$ be a chain such that*:

(1) *$\mathcal{C}$ is a maximal chain of subspaces of $\mathfrak{X}$.*

(2) *Given $\mathfrak{M} \in \mathcal{C}$, there exists $\mathfrak{N} \in \mathcal{C}$, $\mathfrak{N} \neq \mathfrak{M}$, such that either $\mathfrak{M} \subset \mathfrak{N}$ and $\{\mathfrak{M}' \in \mathcal{C}: \mathfrak{M} \subset \mathfrak{M}' \subset \mathfrak{N}\}$ is a well-ordered (from above or from below!) "segment" of $\mathcal{C}$, or $\mathfrak{N} \subset \mathfrak{M}$ and $\{\mathfrak{M}' \in \mathcal{C}: \mathfrak{N} \subset \mathfrak{M}' \subset \mathfrak{N}\}$ is a well-ordered (from above or from below!) "segment" of $\mathcal{C}$.*

Then every clopen subset of $\sigma_\Lambda(T)$ contains a diagonal entry, every component of $\sigma_\Lambda(T)$ intersects $d(T)^-$ and every isolated point of $\sigma_\Lambda(T)$ is a diagonal entry.

PROOF. As in the proofs of Theorems 2 and 2*, it will be enough to show that every clopen subset of $\sigma_\Lambda(T)$ contains a diagonal entry. Let σ_0 be a nonempty clopen subset of $\sigma_\Lambda(T)$. Now observe that Lemma 4 reduces our problem to show that σ_0 contains a diagonal entry of the restriction of T to $\operatorname{ran} E_0$, where E_0 is the idempotent associated to σ_0. In other words: it is enough to prove the result for the case when $\sigma_0 = \sigma_\Lambda(T)$.

Let $\mathfrak{M}, \mathfrak{N}$ be any pair of subspaces in $\mathcal{C}$ such that $\mathfrak{M} \subsetneq \mathfrak{N}$ and the segment $[\mathfrak{M}, \mathfrak{N}]$ is well-ordered, and consider the operator $L =$ the operator induced by $T|\mathfrak{N}$ on $\mathfrak{N}/\mathfrak{M}$. Then (see [5]) $\sigma_\Lambda(L) \subset \sigma_\Lambda(T)$ and Theorems 2 and 2* guarantee that $\sigma_\Lambda(L) \cap d(L) \neq \varnothing$. Since $d(L) \subset d(T)$, the proof is complete. □

3. The case when the restrictions have disjoint spectra.

THEOREM 6. *Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ be a finite family of subspaces invariant under the operator T and assume that $\sigma(T|\mathfrak{M}_k)$ has empty intersection with $\bigcup_{j \neq k} \sigma(T|\mathfrak{M}_j)$ for each $k = 1, \ldots, n$. Then the algebraic sum of the $\mathfrak{M}_j$'s is direct and, moreover, the direct sum $\bigoplus_{j=1}^n \mathfrak{M}_j$ is closed in $\mathfrak{X}$. In addition, if $\bigoplus_{j=1}^n \mathfrak{M}_j = \mathfrak{X}$, then the $\mathfrak{M}_j$'s are actually hyperinvariant subspaces of T and $\sigma(T) = \bigcup_{j=1}^n \sigma(T|\mathfrak{M}_j)$.*

We shall need an auxiliary result. Indeed, the following lemma proves more than what we need for the proof of Theorem 6.

LEMMA 7. *Let $\mathfrak{M}, \mathfrak{N}$ be two invariant subspaces of T and assume that $\sigma(T|\mathfrak{M}) \cap \sigma(T|\mathfrak{N}) = \varnothing$. Assume, moreover, that $\sigma(T|\mathfrak{M}) = \bigcup_{j=1}^k \sigma_{2j-1}$ and $\sigma(T|\mathfrak{N}) = \bigcup_{j=1}^k \sigma_{2j}$, where the σ_{2j-1}'s (σ_{2j}'s, resp.) are clopen subsets of $\sigma(T|\mathfrak{M})$ ($\sigma(T|\mathfrak{N})$, resp.) and that there exist a polynomial $p(z)$ and real constants $\delta_0 < 0 < \delta_1 < \cdots < \delta_{2k}$ such that $\sigma_h \subset \{z: \delta_{h-1} < |p(z)| < \delta_h\}$,*

$h = 1, 2, \ldots, 2k$. *Then* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ *and* $\mathfrak{M} \oplus \mathfrak{N}$ (*algebraic direct sum*) *is closed in* $\mathfrak{X}$.

PROOF. Let $\mathfrak{X}_0 = \mathfrak{M} \cap \mathfrak{N}$; clearly, $\mathfrak{X}_0$ is invariant under T. Thus, according to [1], $\partial\sigma(T|\mathfrak{X}_0) \subset \sigma(T|\mathfrak{M}) \cap \sigma(T|\mathfrak{N}) = \varnothing$; hence, $\sigma(T|\mathfrak{X}_0) = \varnothing$ and, therefore, $\mathfrak{X}_0 = \{0\}$.

By the Riesz functional calculus [10, p. 421], $\mathfrak{M}$ can be written as the algebraic direct sum $\mathfrak{M} = \bigoplus_{j=1}^{k} \mathfrak{M}_{2j-1}$, where $\mathfrak{M}_{2j-1}$ is invariant under T and $\sigma(T|\mathfrak{M}_{2j-1}) = \sigma_{2j-1}$, $j = 1, \ldots, k$. Similarly we have $\mathfrak{N} = \bigoplus_{j=1}^{k} \mathfrak{N}_{2j}$, where $\mathfrak{N}_{2j}$ is invariant under T and $\sigma(T|\mathfrak{N}_{2j}) = \sigma_{2j}$, $j = 1, \ldots, k$.

Let us assume that $\mathfrak{M} \oplus \mathfrak{N}$ is not closed in $\mathfrak{X}$; then (see [9, p. 219]) there exist two sequences of vectors $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $x_n \in \mathfrak{M}$, $y_n \in \mathfrak{N}$, $\|x_n\| = \|y_n\| = 1$, for all n, such that $\lim(n \to \infty)\|x_n - y_n\| = 0$. By using the above decompositions, we can write $x_n = \sum_{j=1}^{k} x_n^{2j-1}$, $y_n = \sum_{j=1}^{k} y_n^{2j}$, where $x_n^{2j-1} \in \mathfrak{M}_{2j-1}$ and $y_n^{2j} \in \mathfrak{N}_{2j}$, $j = 1, \ldots, k$. Moreover, since $\mathfrak{N} = (\bigoplus_{j=1}^{k-1} \mathfrak{N}_{2j}) \oplus \mathfrak{N}_{2k}$, the projection P_{2k} of $\mathfrak{N}$ onto $\mathfrak{N}_{2k}$ along $\bigoplus_{j=1}^{k-1} \mathfrak{N}_{2j}$ is bounded in $\mathfrak{N}$ and therefore there exists a positive constant C_{2k} such that $\|y_n^{2k}\| = \|P_{2k} y_n\| \leqslant C_{2k}$, for all n.

Claim. If $\|x_n - y_n\| \to 0$, then $\|y_n^{2k}\| \to 0$ $(n \to \infty)$. Assume it is not true; then, passing if necessary to a subsequence, we can assume that $\|y_n^{2k}\| \geqslant \varepsilon > 0$, for some ε and for all n.

Let $A = (1/\delta_{2k-1})p(T)$. Clearly, every invariant subspace of T is also invariant under A and, by the spectral mapping theorem (see [10, p. 432]),

$$\sigma(A|\mathfrak{M}) = (1/\delta_{2k-1})p[\sigma(T|\mathfrak{M})] \subset \{z\colon |z| < r < 1\},$$

for some r, $0 < r < 1$,

$$\begin{aligned}\sigma\left(A| \bigoplus_{j=1}^{k-1} \mathfrak{N}_{2j}\right) &= (1/\delta_{2k-1})p\left[\sigma\left(T| \bigoplus_{j=1}^{k-1} \mathfrak{N}_{2j}\right)\right] \\ &\subset \{z\colon |z| < r' < r < 1\},\end{aligned}$$

for some r', $0 < r' < r$, and

$$\begin{aligned}\sigma(A|\mathfrak{N}_{2k}) &= (1/\delta_{2k-1})p[\sigma(T|\mathfrak{N}_{2k})] \\ &= (1/\delta_{2k-1})p(\sigma_{2k}) \subset \{z\colon |z| > R\},\end{aligned}$$

for some $R > 1$.

By using the properties of the spectral radius [10, p. 425], we conclude that, for all m large enough,

$$\|A^m y_n^{2k}\| \geqslant R^m \|y_n^{2k}\| \geqslant \varepsilon R^m \to \infty \qquad (\text{as } m \to \infty),$$

while

$$\begin{aligned}\|A^m(y_n - y_n^{2k} - x_n)\| &\leqslant \|A^m(y_n - y_n^{2k})\| + \|A^m x_n\| \\ &\leqslant (1 + C_{2k})r'^m + r^m \\ &\leqslant (2 + C_{2k})r^m \to 0, \qquad \text{as } m \to \infty.\end{aligned}$$

Since the above estimations do not depend on n, we can fix an m_0 so that $\|A^{m_0}(y_n - x_n)\| \geqslant 1$ for all n; from $\|x_n - y_n\| \to 0$ $(n \to \infty)$, it follows that

$$1 \leqslant \|A^{m_0}(x_n - y_n)\| \leqslant \|A^{m_0}\| \cdot \|x_n - y_n\| \to 0 \qquad (n \to \infty),$$

a contradiction. This proves our claim.

Hence, if $\|x_n - y_n\| \to 0$, then $y_n^{2k} \to 0$ and, for all n large enough, $\|y_n^{2k}\| < \frac{1}{2}$; writing y_n' for $(1/\|y_n - y_n^{2k}\|)\ (y_n - y_n^{2k})$, it follows that $\|x_n - y_n'\| \to 0$ as $n \to \infty$. By repeating the same arguments, it follows that $x_n^{2k-1} \to 0$; then, by induction on k, we conclude that $x_n \to 0$ and $y_n \to 0$, a contradiction. Therefore $\mathfrak{M} \oplus \mathfrak{N}$ is closed in $\mathfrak{X}$. $\square$

Proof of Theorem 6. Our hypothesis on the spectra of the operators $T|\mathfrak{M}_j$, $j = 1, \ldots, n$, is equivalent to saying that there exists a finite family $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ of rectifiable pairwise disjoint closed Jordan curves such that $\sigma(T|\mathfrak{M}_k)$ is separated from $\bigcup_{j \neq k} \sigma(T|\mathfrak{M}_j)$ by some subset of the γ_h's. Clearly, we can assume that no subset of $m - 1$ curves has the desired property (in other words, that Γ is "minimal"); then, after a suitable renumbering of the curves, we can also assume that the *interior* of γ_1 (i.e., the bounded component of $\mathbf{C} \setminus \gamma_1$) does not contain any of the curves $\gamma_2, \ldots, \gamma_m$ and, by induction, that the interior of γ_k does not contain any of the curves $\gamma_{k+1}, \ldots, \gamma_m$, for all $k = 1, 2, \ldots, m - 1$.

Since Γ is minimal, $(\operatorname{int} \gamma_1) \cap \sigma$ (where $\sigma = \bigcup_{j=1}^n \sigma(T|\mathfrak{M}_j)$) is a nonempty clopen subset of one of the $\sigma(T|\mathfrak{M}_j)$'s. Let us assume that $(\operatorname{int} \gamma_1) \cap \sigma = \sigma_1$ is a nonempty clopen subset of $\sigma(T|\mathfrak{M}_1)$; then the Riesz functional calculus gives a decomposition of the form $\mathfrak{M}_1 = \mathfrak{N}_1 \oplus \mathfrak{M}_1'$, where $\mathfrak{N}_1$, $\mathfrak{M}_1'$ are two invariant subspaces of T such that $\sigma(T|\mathfrak{N}_1) = \sigma_1$, $\sigma(T|\mathfrak{M}_1')$ is the (possibly empty) subset $\sigma(T|\mathfrak{M}_1) \setminus \sigma_1$ and the spectra $\sigma(T|\mathfrak{N}_1')$, $\sigma(T|\mathfrak{M}_2), \ldots, \sigma(T|\mathfrak{M}_n)$ are separated by the $m - 1$ curves $\gamma_2, \ldots, \gamma_m$. By induction, we finally obtain a family of invariant subspaces $\mathfrak{N}_1, \ldots, \mathfrak{N}_m$, $\mathfrak{N}_{m+1}$ with nonempty spectra $\sigma_1, \ldots, \sigma_{m+1}$ such that

$$\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_k \subset (\operatorname{int} \gamma_1) \cup (\operatorname{int} \gamma_2) \cup \cdots \cup (\operatorname{int} \gamma_k),$$

for $k = 1, 2, \ldots, m$, and $\bigvee_{h=1}^{m+1} \mathfrak{N}_h = \bigvee_{j=1}^n \mathfrak{M}_j$ (because each of the $\mathfrak{M}_j$'s can be written as a finite direct sum of some of the $\mathfrak{N}_h$'s).

It will be enough to prove that $\bigvee_{h=1}^{m+1} \mathfrak{N}_h = \bigoplus_{h=1}^{m+1} \mathfrak{N}_h$. In fact, if this last equality is true, then

$$\bigvee_{j=1}^{n} \mathfrak{M}_j = \bigvee_{h=1}^{m+1} \mathfrak{N}_h = \bigoplus_{h=1}^{m+1} \mathfrak{N}_h = \bigoplus_{j=1}^{n} \mathfrak{M}_j = \mathfrak{X}_0;$$

each of the $\mathfrak{N}_h$'s (and, a fortiori, each of the $\mathfrak{M}_j$'s) is a hyperinvariant subspace of

$$T|\mathfrak{X}_0 = \bigoplus_{h=1}^{m+1} T|\mathfrak{N}_h = \bigoplus_{j=1}^{n} T|\mathfrak{M}_j$$

and

$$\sigma(T|\mathfrak{X}_0) = \bigcup_{h=1}^{m+1} \sigma_h = \bigcup_{j=1}^{n} \sigma(T|\mathfrak{M}_j)$$

(see [2], [6], [10]).

We shall need a remarkable result due to David Hilbert ([7]; see also [12] for a generalization of this result, partially related with Lemma 7), which asserts that every closed rectifiable Jordan curve can be uniformly approximated by a sequence of *lemniscates* (i.e., level curves of polynomials). Since the distance from γ_h to σ is positive, it is clear that each of the curves γ_h can be replaced by a lemniscate. In other words, we can directly assume that, for each $h = 1, \ldots, m$, there exists a polynomial with complex coefficients, $p_h(z)$, such that $\gamma_h = \{z\colon |p_h(z)| = 1\}$. Thus, if $m = 1$ then the result follows immediately from Lemma 7.

We shall proceed by induction on m. Let us assume that the result is true whenever the spectra can be separated by $m - 1$ curves. This implies, in particular, that $\mathfrak{X}_1 = \bigvee_{h=1}^{m}\mathfrak{N}_h = \bigoplus_{h=1}^{m}\mathfrak{N}_h$, each of the $\mathfrak{N}_h$'s, $h = 1, \ldots, m$, is hyperinvariant for $T|\mathfrak{X}_1 = \bigoplus_{h=1}^{m} T|\mathfrak{N}_h$ and $\sigma(T|\mathfrak{X}_1) = \bigcup_{h=1}^{m}\sigma_h$. Therefore, $\mathfrak{X}_1$ and $\mathfrak{N}_{m+1}$ are invariant subspaces of T whose spectra $\sigma(T|\mathfrak{X}_1)$ and $\sigma(T|\mathfrak{N}_{m+1}) = \sigma_{m+1}$ are separated by a single lemniscate γ_m. Hence, we can apply the results of Lemma 7 to this pair of subspaces to conclude that

$$\mathfrak{X}_0 = \bigvee_{h=1}^{m+1} \mathfrak{N}_h = \mathfrak{X}_1 \vee \mathfrak{N}_{m+1} = \left(\bigoplus_{h=1}^{m} \mathfrak{N}_h\right) \oplus \mathfrak{N}_{m+1} = \bigoplus_{h=1}^{m+1} \mathfrak{N}_h = \bigoplus_{j=1}^{n} \mathfrak{M}_j$$

is closed in $\mathfrak{X}$ and $\sigma(T|\mathfrak{X}_0) = \sigma$. In particular, each of the subspaces $\mathfrak{M}_j$, $j = 1, \ldots, n$, is hyperinvariant for $T|\mathfrak{X}_0$. The proof is now complete. □

REMARK. Example E shows that the hypothesis of Theorem 6 cannot be relaxed.

REFERENCES

1. T. Crimmins and P. Rosenthal, *On the decomposition of invariant subspaces*, Bull. Amer. Math. Soc. **73** (1967), 97–99.

2. R. G. Douglas and C. Pearcy, *On a topology for invariant subspaces*, J. Functional Analysis **2** (1968), 323–341.

3. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967.

4. H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964.

5. D. A. Herrero, *On analytically invariant subspaces and spectra*, Trans. Amer. Math. Soc. **233** (1977), 37–44.

6. D. A. Herrero and N. Salinas, *Analytically invariant and bi-invariant subspaces*, Trans. Amer. Math. Soc. **173** (1972), 117–136.

7. D. Hilbert, *Über die Entwicklung einer beliebigen analytischen Funktion einer Variablen in eine unendliche nach ganzen rationalen Funktionen fortschreitende Reihe*, Gottinger Nachritten (1897), 63–70.

8. T. B. Hoover, *Quasi-similarity of operators*, Illinois J. Math. **16** (1972), 678–686.

9. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.

10. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Ungar, New York, 1955.

11. J. D. Stafney, *The spectrum of certain lower triangular matrices as operators on the l_p spaces*, Pacific J. Math. **42** (1972), 515–526.

12. J. L. Walsh and H. G. Russell, *On the convergence and overconvergence of sequences of polynomials of best simultaneous approximation to several functions in distinct regions*, Trans. Amer. Math. Soc. **36** (1934), 13–28.

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